

GEOMETRY OF METRICS

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ABSTRACT. During the past thirty years hyperbolic type metrics have become popular tools in modern mapping theory, e.g., in the study of quasiconformal and quasiregular maps in the euclidean n -space. We study here several metrics that one way or another are related to modern mapping theory and point out many open problems dealing with the geometry of such metrics.

1. INTRODUCTION

Many results of classical function theory (CFT) are more natural when expressed in terms of the hyperbolic metric than the euclidean metric. Naturality refers here to invariance with respect to conformal maps or specific subgroups of Möbius transformations. For example, one of the corner stones of CFT, the Schwarz lemma [A1], says that an analytic function of the unit disk into itself is a hyperbolic contraction, i.e., decreases hyperbolic distances. Another example is Nevanlinna's principle of the hyperbolic metric [N-53, p. 50]. Note that the hyperbolic metric is invariant under conformal maps. Usual methods of CFT such as power series, integral formulas, calculus of residues, are mainly concerned with the local behavior of functions and do not reflect invariance very well. The extremal length method of Ahlfors and Beurling [AhB] has conformal invariance as a built-in feature and has become a powerful tool of CFT during the past sixty years that have elapsed since its discovery. One may even say that conformal invariance, and thus "naturality", is one of the guiding principles of geometric function theory.

There are serious obstacles in generalizing these ideas from the two-dimensional case to euclidean spaces of dimension $n \geq 3$. For instance, basic facts such as multiplication of complex numbers or power series of functions, do not make sense here. Perhaps a more dramatic obstacle is the failure of Riemann's mapping theorem for dimensions $n \geq 3$: according to Liouville's theorem, conformal maps

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of a domain $D \subset \mathbb{R}^n$ onto $D' \subset \mathbb{R}^n$ are of the form $f = g|D$ for some Möbius transformation g .

Here we shall study various ways to generalize the hyperbolic metric to the n -dimensional case $n \geq 3$. In order to circumvent the above difficulties, we do not require complete invariance with respect to a group of transformations, but only require “quasi-invariance” under transformations called quasi-isometries. Now there are numerous “degrees of freedom”, for instance some of the questions posed below make sense in general metric spaces equipped with some special properties. Therefore the problems below allow for a great number of variations, depending on the particular metric or on the geometry of the space. The Dictionary of Distances [DD-09] lists hundreds of metrics.

This survey is based on my lectures held in two workshops/conferences at IIT-Madras in December 2009 and August 2010. In December 2005 I gave a similar survey [Vu-05] and this survey partially overlaps it. The main difference is that here mainly metric spaces are studied while in the previous survey also categories of maps between metric spaces such as bilipschitz maps or quasiconformal maps were considered. During the past decade the progress has been rapid in this area as shown by the several recent PhD theses [Ha-03, HE, I, Kle-09, Lin-05, Man-08, SA]. In fact, some of the many problems formulated in [Vu-05] have been solved in [Kle-09, Lin-05, Man-08]. The original informal lecture style has been mainly kept without major changes. As in [Vu-05], several problems of varying level of difficulty, from challenging exercises to research problems, are given. It was assumed that the audience was familiar with basic real and complex analysis. The interested reader might wish to study some of the earlier surveys such as [Ge-99, Ge-05, V-99, Vu4, Vu5, Vu-05].

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2. TOPOLOGICAL AND METRIC SPACES

We list some basic notions from topology and metric spaces. For more information on this topic the reader is referred to some standard textbook of topology such as Gamelin and Greene [GG].

The notion of a metric space was introduced by M. Frechét in his thesis in 1906. It became quickly one of the key notions of topology, functional analysis and geometry. In fact, distances and metrics occur in practically all areas of mathematical research, see the book [DD-09]. Modern mapping theory in the setup of metric spaces with some additional structure has been developed by Heinonen [Hei-01].

2.1. Metric space (X, d) . Let X be a nonempty set and let $d : X \times X \rightarrow [0, \infty)$ be a function satisfying

- (a) $d(x, y) = d(y, x)$, for all $x, y \in X$,
- (b) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$,
- (c) $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$.

2.2. Examples.

- (1) $(\mathbb{R}^n, |\cdot|)$ is a metric space.
- (2) If $(X_j, d_j), j = 1, 2$, are metric spaces and $f : (X_1, d_1) \rightarrow (X_2, d_2)$ is an injection, then $m_f(x, y) = d_2(f(x), f(y))$ is a metric.
- (3) If (X, d) is a metric space, then also (X, d^a) is a metric space for all $a \in (0, 1]$.
- (4) More generally, if $h : [0, \infty) \rightarrow [0, \infty)$ is an increasing homeomorphism with $h(0) = 0$ such that $h(t)/t$ is decreasing, then $(X, h \circ d)$ is a metric space (see e.g. [AVV, 7.42]).

2.3. Proposition. Let (X, d) be a metric space, $0 < a \leq 1 \leq b < \infty$, and

$$\rho(x, y) = \max\{d(x, y)^a, d(x, y)^b\}.$$

Then

$$\rho(x, y) \leq 2^{b-1}(\rho(x, z) + \rho(z, y))$$

for all $x, y, z \in X$. In particular, ρ is a metric if $b = 1$.

Proof. Fix $x, y, z \in X$. Consider first the case $d(x, y) \leq 1$. Then

$$\rho(x, y) = d(x, y)^a \leq (d(x, z) + d(z, y))^a \leq d(x, z)^a + d(z, y)^a \leq \rho(x, z) + \rho(z, y),$$

by an elementary inequality [AVV, (1.41)]. Next for the case $d(x, y) \geq 1$ we have

$$\begin{aligned} \rho(x, y) = d(x, y)^b &\leq (d(x, z) + d(z, y))^b \leq 2^{b-1}(d(x, z)^b + d(z, y)^b) \leq \\ &2^{b-1}(\rho(x, z) + \rho(z, y)) \end{aligned}$$

by [AVV, (1.40)]. □

2.4. Remark. If (X, d) is a metric space and ρ is as defined above in 2.3, then by a result of A. H. Frink [F], there is a metric d_1 such that $d_1 \leq \rho^{1/k} \leq 4d_1$. This result was recently refined by M. Paluszynski and K. Stempak [PS-09]. I am indebted to J. Luukkainen for this remark.

2.5. Uniform continuity. Let $(X_j, d_j), j = 1, 2$, be metric spaces and $f : (X_1, d_1) \rightarrow (X_2, d_2)$ be a continuous map. Then f is uniformly continuous (u.c.) if there exists a continuous injection $\omega : [0, t_0) \rightarrow [0, \infty)$ such that $\omega(0) = 0$ and

$$d_2(f(x), f(y)) \leq \omega(d_1(x, y)), \text{ for all } x, y \in X_1 \text{ with } d_1(x, y) < t_0.$$

2.6. Remarks.

- (1) This definition is equivalent with the usual (ε, δ) -definition [GG].
- (2) If $\omega(t) = Lt$ for $t \in (0, t_0]$, then f is L -Lipschitz (abbr. L -Lip).
- (3) If $\omega(t) = Lt^a$ for some $a \in (0, 1]$ and all $t \in (0, t_0]$, then f is Hölder.
- (4) If $f : (X_1, d_1) \rightarrow (X_2, d_2)$ is a bijection and there is $L \geq 1$ such that $d_1(x, y)/L \leq d_2(f(x), f(y)) \leq Ld_1(x, y)$ for all $x, y \in X_1$ then f is L -bilipschitz. Sometimes bilipschitz maps are also called quasi-isometries.
- (5) A map is said to be an isometry if it is 1-bilipschitz.
- (6) The map $f : (X, |\cdot|) \rightarrow (X, |\cdot|)$, $X = (0, \infty)$, $f(x) = 1/x$, for $x \in X$ is not uniformly continuous. We will later see that this map is uniformly continuous with respect to the hyperbolic metric of X .
- (7) A Lip map $h : [a, b] \rightarrow \mathbb{R}$ has a derivative a.e.

2.7. Balls. Write $B_d(x_0, r) = \{x \in X : d(x_0, x) < r\}$ and $\overline{B}_d(x_0, r) = \{x \in X : d(x_0, x) \leq r\}$.

2.8. Fact. Let $\tau = \{B_d(x, r) : x \in X, r > 0\}$ be the collection of all balls. Then $(X, \tau \cup \{\emptyset\} \cup \{X\})$ is a topology.

2.9. Remarks.

- (1) We always equip a metric space with this topology.
- (2) The balls $\overline{B}_d(x_0, r)$ and $B_d(x_0, r)$ are closed and open as point sets, resp.
- (3) The set (\mathbb{Z}, d) , $d(x, y) = |x - y|$ is a metric space. Then $B_d(0, 1) = \{0\}$, $\overline{B}_d(0, 1) = \{-1, 0, 1\}$. Hence $\text{clos}(B_d(x_0, r))$ need not be $\overline{B}_d(x_0, r)$. Also $\text{diam}(B_d(0, 1)) = 0 < \text{diam}(\overline{B}_d(0, 1)) = 2$.
- (4) Balls in \mathbb{R}^n need not be connected (cf. below).
- (5) In \mathbb{R}^n : balls are denoted by $B^n(x, r)$ and spheres by $\partial B^n(x, r) = S^{n-1}(x, r)$.

2.10. Paths. A continuous map $\gamma : \Delta \rightarrow X$, $\Delta \subset \mathbb{R}$, is called a path. The length of γ , $\ell(\gamma)$, is

$$\ell(\gamma) = \sup \left\{ \sum_{i=1}^n d(\gamma(x_{i-1}), \gamma(x_i)) : \{x_0, \dots, x_n\} \text{ is a subdivision of } \Delta \right\}.$$

We say a path is rectifiable if $\ell(\gamma) < \infty$. A rectifiable path $\gamma : \Delta \rightarrow X$ has a parameterization in terms of arc length $\gamma^o : [0, \ell(\gamma)] \rightarrow X$.

2.11. Definition. A set G is connected if for all $x, y \in G$ there exists a path $\gamma : [0, 1] \rightarrow G$ such that $\gamma(0) = x$, $\gamma(1) = y$. Sometimes we write Γ_{xy} for the set of all paths joining x with y in G .

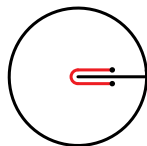


FIGURE 1. A non-convex set which has no Euclidean geodesics

2.12. Inner metric of a set $G \subset X$. For fixed $x, y \in X$ the inner metric with respect to G is defined by $d(x, y) = \inf\{\ell(\gamma) : \gamma \in \Gamma_{xy}, \gamma \subset G\}$.

2.13. Geodesics. A path $\gamma : [0, 1] \rightarrow G$ where G is a domain, is a geodesic joining $\gamma(0)$ and $\gamma(1)$ if $\ell(\gamma) = d(\gamma(0), \gamma(1))$ and $d(\gamma(0), \gamma(t)) + d(\gamma(t), \gamma(1)) = \ell(\gamma)$ for all $t \in (0, 1)$.

2.14. Remarks.

- (1) In $(\mathbb{R}^n, |\cdot|)$ the segment $[x, y] = \{z \in \mathbb{R}^n : z = \lambda x + (1 - \lambda)y, \lambda \in [0, 1]\}$ is a geodesic.
- (2) Let $G = B^2 \setminus \{0\}$ and d be the inner metric of G . There are no geodesics joining $-1/2$ and $1/2$ in (G, d) .

2.15. Problems. ([Vu-05, p. 322]) Let X be a locally convex set in \mathbb{R}^n and let (X, d) be a metric space.

- (1) When are balls $B_d(x, t)$ convex for all radii $t > 0$?
- (2) When are balls convex for small radii t ?
- (3) When are the boundaries of balls nice/smooth?

2.16. Ball inclusion problem. Suppose that $(X, d_j), j = 1, 2$, determine the same euclidean topology, $X \subset \mathbb{R}^n$. Then

$$B_{d_1}(x_0, r) \subset B_{d_2}(x_0, s) \subset B_{d_1}(x_0, t)$$

for some $r, s, t > 0$. For a fixed $s > 0$, find the best radii r and t . This problem is interesting and open for instance for several pairs of the metrics d_1, d_2 even in the special case when one of the metrics is the euclidean metric.

Note that in the above problems 2.15 and 2.16 we consider the general metric space situation. It is natural to expect that useful answers can only be given under additional hypotheses. The reader is encouraged to find such hypotheses. In some particular cases the problems will be studied below.

3. PRINCIPLES OF GEOMETRY

In this section we continue our list of metrics and introduce some necessary terminology. We also outline the principles of geometry, according to F. Klein. These principles provide a uniform view of various geometries. In particular, the basic models of geometry: the euclidean geometry, the geometry of the Riemann sphere and the hyperbolic geometry of the unit ball fit into this framework.

The search for geometries leads us to compare the properties of geometries. The Klein principles, known under the name "Erlangen Program", have also paved the road for the development of geometric function theory during the past century. For a broad review of basic and advanced geometry we recommend [Ber-87] and [BBI-01].

3.1. Path integrals. For a locally rectifiable path $\gamma : \Delta \rightarrow X$ and a continuous function $f : \gamma\Delta \rightarrow [0, \infty]$, the path integral is defined in two steps. Recall that γ^o is the normal representation of a rectifiable path.

[I] If γ is rectifiable, we set

$$\int_{\gamma} f ds = \int_0^{\ell(\gamma)} f(\gamma^o(t)) |(\gamma^o)'(t)| dt.$$

[II] If γ is locally rectifiable, we set

$$\int_{\gamma} f ds = \sup \left\{ \int_{\beta} f ds : \ell(\beta) < \infty, \beta \text{ is a subpath of } \gamma \right\}.$$

3.2. Weighted length. Let $G \subset X$ be a domain and $w : G \rightarrow (0, \infty)$ continuous. For fixed $x, y \in D$, define

$$d_w(x, y) = \inf \{ \ell_w(\gamma) : \gamma \in \Gamma_{xy}, \ell(\gamma) < \infty \}, \ell_w(\gamma) = \int_{\gamma} w(\gamma(z)) |dz|.$$

It is easy to see that d_w defines a metric on G and (G, d_w) is a metric space. If a length-minimizing curve exists, it is called a geodesic.

The above construction of the weighted length 3.2 has many applications in geometric function theory. For instance the hyperbolic and spherical metrics are special cases of it. Our first example of 3.2 is the quasihyperbolic metric, which has been recently studied by numerous authors. See for instance the papers [KM] and [KRT] in this proceedings and their bibliographies.

3.3. Quasihyperbolic metric. If $w(x) = 1/d(x, \partial G)$, then d_w is the quasihyperbolic metric of a domain $G \subset \mathbb{R}^n$. Gehring and Osgood have proved [GO-79] that geodesics exist in this case. Note that $w(x) = 1/d(x, \partial G)$ is like a "penalty-function", the geodesic segments try to keep away from the boundary.

3.4. Examples.

- (1) If $G = \mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$ then the quasihyperbolic metric coincides with the usual hyperbolic metric, to be discussed later on. Often the notation $\rho_{\mathbb{H}^n}$ is used.
- (2) The hyperbolic metric of the unit ball \mathbb{B}^n is a weighted metric with the weight function $w(x) = 2/(1 - |x|^2)$. Often the notation $\rho_{\mathbb{B}^n}$ is used.
- (3) In the special case when $w \equiv 1$ and the domain G is a convex subdomain $G \subset \mathbb{R}^n$, d_w is the Euclidean distance. The geodesics are the Euclidean segments.
- (4) In the special case when $w \equiv 1$ in the non-convex set $G = B^2 \setminus [0, 1)$ geodesics do not exist (consider the points $a = \frac{1}{2} + \frac{i}{10}$ and $b = \bar{a}$). See Fig. 1.
- (5) If the construction 3.2 is applied to \mathbb{R}^n with the weight function $1/(1 + |x|^2)$ we obtain the spherical metric. This spaces can be identified with the Riemann sphere $S^n(e_{n+1}/2, 1/2)$ equipped with the usual arc-length metric.
- (6) Let $X = \{x \in \mathbb{R} : x > 0\}$ and $w(x) = 1/x, x \in X$. Then we see that $\ell_w(x, y) = |\log(x/y)|$ for all $x, y \in X$. Consider again the map $f : X \rightarrow X, f(x) = 1/x, x \in X$. We have seen in 2.6(6) that it is not uniformly continuous. But it is uniformly continuous as a map $f : (X, \ell_w) \rightarrow (X, \ell_w)$.

3.5. The Möbius group $\mathcal{GM}(\overline{\mathbb{R}^n})$. The group of Möbius transformations in $\overline{\mathbb{R}^n}$ is generated by transformations of two types

- (1) inversions in $S^{n-1}(a, r) = \{z \in \mathbb{R}^n : |a - z| = r\}$

$$x \mapsto a + \frac{r^2(x - a)}{|x - a|^2},$$

- (2) reflections in hyperplane $P(a, t) = \{x \in \mathbb{R}^n : x \cdot a = t\}$

$$x \mapsto x - 2(x \cdot a - t) \frac{a}{|a|^2}.$$

If $G \subset \overline{\mathbb{R}^n}$ we denote by $\mathcal{GM}(G)$ the group of all Möbius transformations with $fG = G$. The stereographic projection $\pi : \overline{\mathbb{R}^n} \rightarrow S^n((1/2)e_{n+1}, 1/2)$ is defined

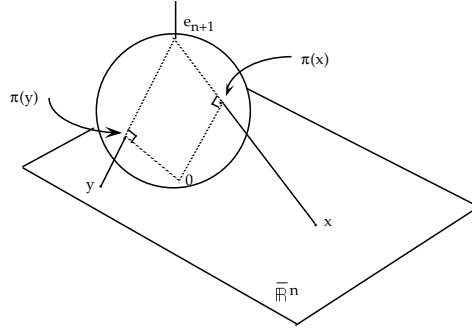


FIGURE 2. Stereographic projection

by a Möbius transformation an inversion in $S^n(e_{n+1}, 1)$:

$$\pi(x) = e_{n+1} + \frac{x - e_{n+1}}{|x - e_{n+1}|^2}, x \in \mathbb{R}^n, \pi(\infty) = e_{n+1}.$$

3.6. Plane versus space.

- (1) For $n = 2$ Möbius transformations are of the form $\frac{az+b}{cz+d}$, $z, a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$.
- (2) Recall that for $n = 2$ there are many conformal maps (Riemann mapping Theorem., Schwarz-Christoffel formula). In contrast for $n \geq 3$ conformal maps are, by Liouville's theorem (suitable smoothness required), Möbius transformations.
- (3) Therefore conformal invariance for the space $n \geq 3$ is very different from the plane case $n = 2$.

3.7. Chordal metric. Stereographic projection defines the chordal distance by

$$q(x, y) = |\pi x - \pi y| = \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}}$$

for $x, y \in \overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$. Perhaps the shortest proof of the triangle inequality for q follows if we use 2.6(2).

3.8. Absolute (cross) ratio. For distinct points $a, b, c, d \in \overline{\mathbb{R}^n}$ the absolute ratio is

$$|a, b, c, d| = \frac{q(a, c)q(b, d)}{q(a, b)q(c, d)}.$$

The most important property is Möbius invariance: if f is a Möbius transformation, then $|fa, fb, fc, fd| = |a, b, c, d|$. Permutations of a, b, c, d lead to 6 different numerical values of the absolute ratio.

3.9. Conformal mapping. If $G_1, G_2 \subset \mathbb{R}^n$ are domains and $f: G_1 \rightarrow G_2$ is a diffeomorphism with $|f'(x)h| = \underbrace{|f'(x)|}_{\text{operator } n. \text{ vector } n.} \underbrace{|h|}_{\text{vector } n.}$ we call f a *conformal map*. We

use this also in the case $G_1, G_2 \subset \mathbb{R}^n$ by excluding the two points $\{\infty, f^{-1}(\infty)\}$.

For instance, Möbius transformations are examples of conformal maps for all dimensions $n \geq 3$ (cf. 3.6 02)

3.10. Linear dilatation. Let $f: (X, d_1) \rightarrow (Y, d_2)$ be a homeomorphism, $x_0 \in X$. We define the linear dilatation $H(x_0, f)$ as follows

$$H(x_0, f, r) = \frac{L_r}{l_r}, \quad H(x_0, f) = \limsup_{r \rightarrow 0} H(x_0, f, r)$$

3.11. Quasiconformal maps. We adopt the definition of Väisälä [V1] for K -quasiconformal (qc) mappings. Recall that for a K -qc, $K \geq 1$, homeomorphism $f: G \rightarrow G', G, G' \subset \mathbb{R}^n$ there exists a constant $H_n(K)$ such that $\forall x_0 \in G$ $H(x_0, f) \leq H_n(K)$. The reader is referred to [V1] and [Ge-05] for the basic properties of quasiconformal maps.

It is well-known that conformal maps are 1-qc. It can be also proved that $H_n(1) = 1$, for the somewhat tedious details, see [AVV].

3.12. Examples. In most examples below, the metric spaces will have additional structure. In particular, we will study metric spaces (X, d) where the group Γ of automorphisms of X acts transitively (i.e. given $x, y \in X$ there exists $h \in \Gamma$ such that $hx = y$). We say that the metric d is quasiinvariant under the action of Γ if there exists $C \in [1, \infty)$ such that $d(hx, hy)/d(x, y) \in [1/C, C]$ for all $x, y \in X, x \neq y$, and all $h \in \Gamma$. If $C = 1$, then we say that d is invariant.

- (1) The Euclidean space \mathbb{R}^n equipped with the usual metric $|x - y| = (\sum_{j=1}^n (x_j - y_j)^2)^{1/2}$, Γ is the group of translations.
- (2) The unit sphere $S^n = \{z \in \mathbb{R}^{n+1} : |z| = 1\}$ equipped with the metric of \mathbb{R}^{n+1} and Γ is the set of rotations of S^n .

3.13. F.Klein's Erlangen Program 1872 for geometry.

- use isometries ("rigid motions") to study geometry
- Γ is the group of isometries
- two configurations are considered equivalent if they can be mapped onto each other by an element of Γ
- the basic "models" of geometry are
 - (a): Euclidean geometry of \mathbb{R}^n
 - (b): hyperbolic geometry of the unit ball B^n in \mathbb{R}^n

(c): spherical geometry (Riemann sphere)

The main examples of Γ are subgroups of Möbius transformations of $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$.

3.14. Example: Rigid motions and invariant metrics.

X	Γ	metric
G	$\mathcal{M}(G)$	ρ_G hyperbolic metric, $G = B^n, \mathbb{H}^n$
$\overline{\mathbb{R}}^n$	$\text{Iso}(\overline{\mathbb{R}}^n)$	q chordal metric
\mathbb{R}^n	transl.	$ \cdot $ Euclidean metric

3.15. **Beyond Erlangen, dictionary of the quasiworld.** For the purpose of studying mappings defined in subdomains of \mathbb{R}^n , we must go beyond Erlangen, to the quasiworld, in order to get a rich class of mappings.

Conformal	\rightarrow	"Quasiconformal"
Invariance	\rightarrow	"Quasi-invariance"
Unit ball	\rightarrow	"Classes of domains"
Metric	\rightarrow	"Deformed metric"
World	\rightarrow	"Quasiworld"
Smooth	\rightarrow	"Nonsmooth"
Hyperbolic	\rightarrow	"Neohyperbolic"

4. CLASSICAL GEOMETRIES

In this section we discuss some basic facts about the hyperbolic geometry, already defined in Section 3 as a particular case of the weighted metric. Some standard sources are [A2, A, Be-82, KL-07]. See also [Vu-88]. We begin by describing the hyperbolic balls in terms of euclidean balls. In passing we remark that this description will provide, in one concrete case, a complete solution to the ball inclusion problem 2.16.

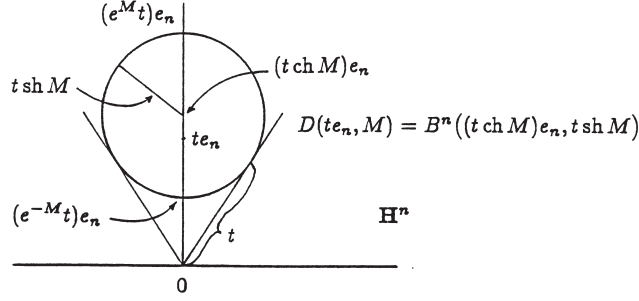
4.1. **Comparison of metric balls.** For $r, s > 0$ we obtain the formula

$$(4.1) \quad \rho(re_n, se_n) = \left| \int_s^r \frac{dt}{t} \right| = \left| \log \frac{r}{s} \right|.$$

Here $e_n = (0, \dots, 1) \in \mathbb{R}^n$. We recall the invariance property:

$$(4.2) \quad \rho(x, y) = \rho(f(x), f(y)).$$

For $a \in \mathbb{H}^n$ and $M > 0$ the *hyperbolic ball* $\{x \in \mathbb{H}^n : \rho(a, x) < M\}$ is denoted by $D(a, M)$. It is well known that $D(a, M) = B^n(z, r)$ for some z and


 FIGURE 3. The hyperbolic ball $D(te_n, M)$ as a Euclidean ball.

r (this also follows from (4.2)!). This fact together with the observation that $\lambda te_n, (t/\lambda)e_n \in \partial D(te_n, M)$, $\lambda = e^M$ (cf. (4.1)), yields

$$(4.3) \quad \begin{cases} D(te_n, M) = B^n((t \cosh M)e_n, t \sinh M) , \\ B^n(te_n, rt) \subset D(te_n, M) \subset B^n(te_n, Rt) , \\ r = 1 - e^{-M} , \quad R = e^M - 1 . \end{cases}$$

It is well known that the balls $D(z, M)$ of (\mathbb{B}^n, ρ) are balls in the Euclidean geometry as well, i.e. $D(z, M) = B^n(y, r)$ for some $y \in \mathbb{B}^n$ and $r > 0$. Making use of this fact, we shall find y and r . Let L_z be a Euclidean line through 0 and z and $\{z_1, z_2\} = L_z \cap \partial D(z, M)$, $|z_1| \leq |z_2|$. We may assume that $z \neq 0$ since with obvious changes the following argument works for $z = 0$ as well. Let $e = z/|z|$ and $z_1 = se$, $z_2 = ue$, $u \in (0, 1)$, $s \in (-u, u)$. Then it follows that

$$\begin{aligned} \rho(z_1, z) &= \log\left(\frac{1+|z|}{1-|z|} \cdot \frac{1-s}{1+s}\right) = M , \\ \rho(z_2, z) &= \log\left(\frac{1+u}{1-u} \cdot \frac{1-|z|}{1+|z|}\right) = M \end{aligned}$$

Solving these for s and u and using the fact that

$$D(z, M) = B^n\left(\frac{1}{2}(z_1 + z_2), \frac{1}{2}|u - s|\right)$$

one obtains the following formulae:

$$(4.4) \quad \begin{cases} D(x, M) = B^n(y, r) \\ y = \frac{x(1-t^2)}{1-|x|^2 t^2} , \quad r = \frac{(1-|x|^2)t}{1-|x|^2 t^2} , \quad t = \tanh \frac{1}{2}M , \end{cases}$$

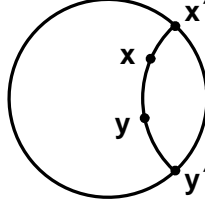


FIGURE 4. Hyperbolic lines are circular arcs perpendicular to ∂B^n and $\rho_{B^n}(x, y) = \log |x', x, y, y'|$.

and

$$\begin{cases} B^n(x, a(1 - |x|)) \subset D(x, M) \subset B^n(x, A(1 - |x|)) , \\ a = \frac{t(1 + |x|)}{1 + |x|t} , \quad A = \frac{t(1 + |x|)}{1 - |x|t} , \quad t = \tanh \frac{1}{2}M . \end{cases}$$

A special case of (4.4):

$$D(0, M) = B_\rho(0, M) = B^n(\tanh \frac{1}{2}M) .$$

For a given pair of points $x, y \in \mathbb{R}^n$ and a number $t > 0$, an Apollonian sphere is the set of all points z such that $|z - x|/|z - y| = t$. It is easy to show that, given $x \in B^n$, hyperbolic spheres with hyperbolic center x are Apollonian spheres w.r.t. the points $x, x/|x|^2$, see [KV1].

Note that balls in chordal metric can be similarly described in terms of the euclidean balls, see [AVV].

4.2. Hyperbolic metric of the unit ball B^n . Four equivalent definitions of the hyperbolic metric ρ_{B^n} .

- (1) $\rho_{B^n} = m_w$, $w(x) = \frac{2}{1 - |x|^2}$.
- (2) $\sinh^2 \frac{\rho_{B^n}(x, y)}{2} = \frac{|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}$.
- (3) $\rho_{B^n}(x, y) = \sup\{\log |a, x, y, d| : a, d \in \partial B^n\}$.
- (4) $\rho_{B^n}(x, y) = \log |x_*, x, y, y_*|$.

The hyperbolic metric is invariant under the action of $\mathcal{GM}(B^n)$, i.e. $\rho(x, y) = \rho(h(x), h(y))$ for all $x, y \in B^n$ and all $h \in \mathcal{GM}(B^n)$.

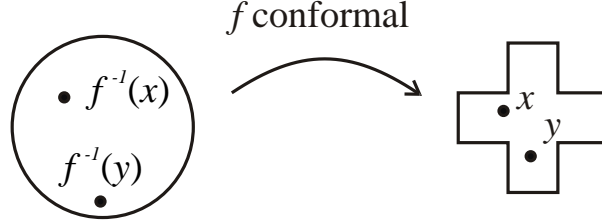


FIGURE 5. Definition of $\rho(x, y)$ in terms of $\rho_{B^n}(f^{-1}(x), f^{-1}(y))$, f conformal.

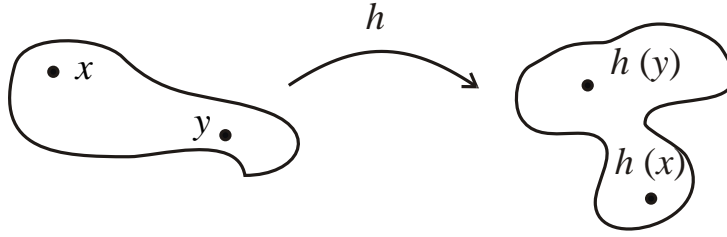


FIGURE 6. Invariance of the hyperbolic metric under conformal map.

4.3. The hyperbolic line through x, y . The hyperbolic geodesics between x, y in the unit ball are the circular arcs joining x and y orthogonal to ∂B^n .

4.4. Hyperbolic metric of $G = f(B^2)$, f conformal. In the case where $G_k = f_k(B^2)$ and f_k is conformal, $k = 1, 2$, it follows that if $h: G_1 \rightarrow G_2$ is conformal, then the hyperbolic metric is invariant under h , i.e., $\rho_{G_1}(x, y) = \rho_{G_2}(hx, hy)$. Thus we may use explicit conformal maps to evaluate the hyperbolic metrics in cases where such a map is known.

For $n = 2$ one can generalize the hyperbolic metric, using covering transformations, to a domain $G \subset \overline{\mathbb{R}^2}$ with $\text{card}(\overline{\mathbb{R}^2} \setminus G) \geq 3$ [KL-07].

The formula for the hyperbolic metric of the unit ball given by 4.2(2) is relatively complicated. Therefore various comparison functions have been introduced. We will now discuss two of them.

4.5. The distance ratio metric j_G . For $x, y \in G$ the *distance ratio metric* j_G is defined [Vu-85] by

$$j_G(x, y) = \log \left(1 + \frac{|x - y|}{\min\{d(x), d(y)\}} \right).$$

The inequality $j_G \leq \delta_G \leq \tilde{j}_G \leq 2j_G$ holds for every open set $G \subsetneq \mathbb{R}^n$, where the metric \tilde{j}_G (cf. [GO-79]) is a metric defined by

$$\tilde{j}_G(x, y) = \log \left(1 + \frac{|x - y|}{d(x)} \right) \left(1 + \frac{|x - y|}{d(y)} \right).$$

We collect the following well-known facts:

- (1) Inner metric of the j_G metric is the quasihyperbolic metric k_G .
- (2) $k_G(x, y) \leq 2j_G(x, y)$ for all $x, y \in G$ with $j_G(x, y) < \log(3/2)$, see [Vu-88, 3.7(2)].
- (3) Both k_G and j_G define the Euclidean topology.
- (4) j_G is not geodesic; the balls $B_j(z, M) = \{x \in G : j_G(z, x) < M\}$ may be disconnected for large M .

If we compare the density functions of the hyperbolic and the quasihyperbolic metrics of B^n , it will lead to the observations that

$$(4.5) \quad \rho_{B^n}(x, y)/2 \leq k_{B^n}(x, y) \leq \rho_{B^n}(x, y)$$

for all $x, y \in B^n$.

The following proposition gathers together several basic properties of the metrics k_G and j_G , see for instance [GP-76, Vu-88].

4.6. Proposition. ([KSV-09])

- (1) For a domain $G \subset \mathbb{R}^n$, $x, y \in G$, and with $L = \inf\{\ell(\gamma) : \gamma \in \Gamma_{x,y}\}$, we have

$$k_G(x, y) \geq \log \left(1 + \frac{L}{\min\{\delta(x), \delta(y)\}} \right) \geq j_G(x, y).$$

- (2) For $x \in B^n$ we have

$$k_{B^n}(0, x) = j_{B^n}(0, x) = \log \frac{1}{1 - |x|}.$$

- (3) Moreover, for $b \in S^{n-1}$ and $0 < r < s < 1$ we have

$$k_{B^n}(br, bs) = j_{B^n}(br, bs) = \log \frac{1 - r}{1 - s}.$$

- (4) Let $G \subsetneq \mathbb{R}^n$ be any domain and $z_0 \in G$. Let $z \in \partial G$ be such that $\delta(z_0) = |z_0 - z|$. Then for all $u, v \in [z_0, z]$ we have

$$k_G(u, v) = j_G(u, v) = \left| \log \frac{\delta(z_0) - |z_0 - u|}{\delta(z_0) - |z_0 - v|} \right| = \left| \log \frac{\delta(u)}{\delta(v)} \right|.$$

- (5) For $x, y \in B^n$ we have

$$j_{B^n}(x, y) \leq \rho_{B^n}(x, y) \leq 2j_{B^n}(x, y)$$

with equality on the right hand side when $x = -y$.

- (6) For $0 < s < 1$ and $x, y \in B^n(s)$ we have

$$j_{B^n}(x, y) \leq k_{B^n}(x, y) \leq (1 + s) j_{B^n}(x, y).$$

Proof. (1) Without loss of generality we may assume that $\delta(x) \leq \delta(y)$. Fix $\gamma \in \Gamma(x, y)$ with arc length parameterization $\gamma : [0, u] \rightarrow G, \gamma(0) = x, \gamma(u) = y$

$$\begin{aligned} \ell_k(\gamma) &= \int_0^u \frac{|\gamma'(t)| dt}{d(\gamma(t), \partial G)} \geq \int_0^u \frac{dt}{\delta(x) + t} = \log \frac{\delta(x) + u}{\delta(x)} \\ &\geq \log \left(1 + \frac{|x - y|}{\delta(x)} \right) = j_G(x, y). \end{aligned}$$

- (2) We see by (1) that

$$j_{B^n}(0, x) = \log \frac{1}{1 - |x|} \leq k_{B^n}(0, x) \leq \int_{[0, x]} \frac{|dz|}{\delta(z)} = \log \frac{1}{1 - |x|}$$

and hence $[0, x]$ is the k_{B^n} -geodesic between 0 and x and the equality in (2) holds.

The proof of (3) follows from (2) because the quasihyperbolic length is additive along a geodesic

$$k_{B^n}(0, bs) = k_{B^n}(0, br) + k_{B^n}(br, bs).$$

The proof of (4) follows from (3).

The proof of (5) is given in [AVV, Lemma 7.56].

For the proof of the last statement see [KSV-09]. □

In view of (4.5) and Proposition 4.6 we see that for the case of the unit ball, the metrics j, k, ρ are closely related.

5. METRICS IN PARTICULAR DOMAINS: UNIFORM, QUASIDISKS

We have seen above that for the case of the unit ball, several metrics are equivalent. This leads to the general question: Suppose that given a domain $G \subset \mathbb{R}^n$ we have two metrics d_1, d_2 on G . Can we characterize those domains G , which for a fixed constant $c > 0$ satisfy $d_1(x, y) \leq c d_2(x, y)$ for all $x, y \in G$. As far as we know, this is a largely open problem. However, the class of domains characterized by the property that the quasihyperbolic and the distance ratio metric have a bounded quotient, coincides with the very widely known class of uniform domains introduced by Martio and Sarvas [MS-79].

There is more general class of domains, so called φ -uniform domains, which contain the uniform domains as special case which we will briefly discuss [Vu-85].

It is easy to see that for a general domain the quasihyperbolic and distance ratio metrics both define the euclidean topology, in fact we can solve the ball inclusion problem 2.16 easily, see [Vu-88, (3.9)] for the case of the quasihyperbolic metric. Although some progress has been made on this problem recently in [KV2], the problem is not completely solved in the case of metrics considered in this survey.

5.1. Uniform domains and constant of uniformity. The following form of the definition of the uniform domain is due to Gehring and Osgood [GO-79]. As a matter of fact, in [GO-79] there was an additive constant in the inequality (5.1), but it was shown in [Vu-85, 2.50(2)] that the constant can be chosen to be 0.

5.2. Definition. A domain $G \subsetneq \mathbb{R}^n$ is called uniform, if there exists a number $A \geq 1$ such that

$$(5.1) \quad k_G(x, y) \leq A j_G(x, y)$$

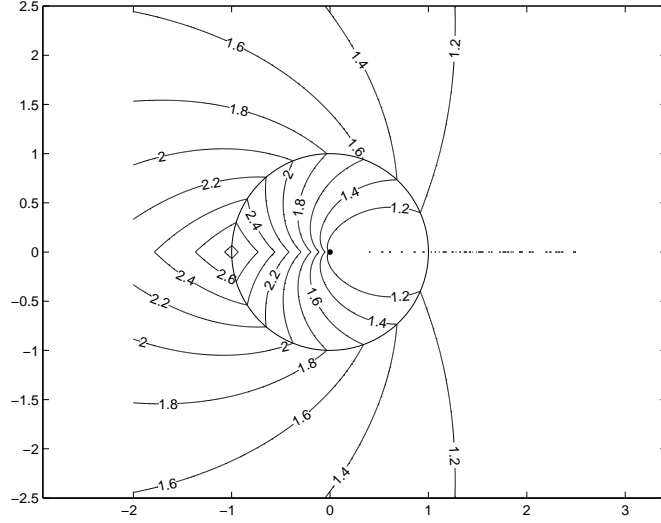
for all $x, y \in G$. Furthermore, the best possible number

$$A_G := \inf\{A \geq 1 : A \text{ satisfies (5.1)}\}$$

is called the uniformity constant of G .

Our next goal is to explore domains for which the uniformity constant can be evaluated or at least estimated. For that purpose we consider some simple domains.

5.3. Examples of quasihyperbolic geodesics.


 FIGURE 7. Sets $\{z : k_G(1, z)/j_G(1, z) = c\}$.

- (1) For the domain $\mathbb{R}^n \setminus \{0\}$ Martin and Osgood (see [MO-86]) have determined the geodesics. Their result states that given $x, y \in \mathbb{R}^n \setminus \{0\}$, the geodesic segment can be obtained as follows: let φ be the angle between the segments $[0, x]$ and $[0, y]$, $0 < \varphi < \pi$. The triple $0, x, y$ clearly determines a 2-dimensional plane Σ , and the geodesic segment connecting x to y is the logarithmic spiral in Σ with polar equation

$$r(\omega) = |x| \exp\left(\frac{\omega}{\varphi} \log \frac{|y|}{|x|}\right).$$

In the punctured space the quasihyperbolic distance is given by the formula

$$k_{\mathbb{R}^n \setminus \{0\}}(x, y) = \sqrt{\varphi^2 + \log^2 \frac{|x|}{|y|}}.$$

- (2) [Lin-05] Let $\varphi \in (0, \pi]$ and $x, y \in S_\varphi = \{(r, \theta) \in \mathbb{R}^2 : 0 < \theta < \varphi\}$, the angular domain. Then the quasihyperbolic geodesic segment is a curve consisting of line segments and circular arcs orthogonal to the boundary. If $\varphi \in (\pi, 2\pi)$, then the geodesic segment is a curve consisting of pieces of three types: line segments, arcs of logarithmic spirals and circular arcs orthogonal to the boundary.
- (3) [Lin-05] In the punctured ball $\mathbb{B}^n \setminus \{0\}$, the quasihyperbolic geodesic segment is a curve consisting of arcs of logarithmic spirals and geodesic segments of the quasihyperbolic metric of B^n .

The above formula 5.3(1) shows that the quasihyperbolic metric of $G = \mathbb{R}^n \setminus \{0\}$ is invariant under the inversion $x \mapsto x/|x|^2$ which maps G onto itself. It is also easy to see that for this domain G also j_B has the same invariance property. Next, for this domain G and a given number $c > 1$, the sets $\{x : k_G(1, x)/j_G(1, x) = c\}$ are illustrated. The invariance under the inversion is quite apparent. The same formula 5.3 (1) is also discussed in [KM].

We now give a list of constants of uniformity for a few specific domains following H. Lindén [Lin-05].

- (1) For the domain $\mathbb{R}^n \setminus \{0\}$, the uniformity constant is given by (cf. Figure 7)

$$A_{\mathbb{R}^n \setminus \{0\}} = \pi / \log 3 \approx 2.8596.$$

- (2) Constant of uniformity in the punctured ball $B^n \setminus \{0\}$ is same as that in $\mathbb{R}^n \setminus \{0\}$.

- (3) For the angular domain S_φ , the uniformity constant is given by

$$A_{S_\varphi} = \frac{1}{\sin \frac{\varphi}{2}} + 1$$

when $\varphi \in (0, \pi]$.

There are numerous characterizations of quasidisks, i.e. quasiconformal images of the unit disk under a quasiconformal map. E.g. it is known that a simply connected domain is a quasidisk if and only if it is a uniform domain, see [Ge-99].

5.4. φ -uniform domains ([Vu-85]). Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a homeomorphism. We say that a domain $G \subset \mathbb{R}^n$ is φ -uniform if

$$k_G(x, y) \leq \varphi(|x - y| / \min\{d(x, \partial G), d(y, \partial G)\})$$

holds for all $x, y \in G$.

In [Vu-85] φ -uniform domains were introduced for the purpose of finding a wide class of domains where various conformal invariants could be compared to each other. Obviously, uniform domains form a subclass. Recently, many examples of these domains were given in [KSV-09]. This class of domains is relative little investigated and there are many interesting questions even in the case of plane simply connected φ -uniform domains. This class of plane domains contains e.g. all quasicircles. Because for a quasicircle C the both components of $\mathbb{C} \setminus C$ are quasidisks, we could ask the following question. Suppose that C is a Jordan curve in the plane dividing thus $\mathbb{C} \setminus C$ into two components, one of which is a φ -uniform domain. Is it true that also the other component is a φ_1 -uniform domain for some function φ_1 ? This question was recently answered in the negative in [HKSV-09].

5.5. Open problem. Is it true that there are φ -uniform domains G in the plane such that the Hausdorff-dimension of ∂G is two?

Recall that for quasicircles this is not possible by [GV]. P. Koskela has informed the author that Tomi Nieminen has done some work on this problem.

6. HYPERBOLIC TYPE GEOMETRIES

In this section we discuss briefly two metrics, the Apollonian metric α_G and a Möbius invariant metric δ_G introduced by P. Seittenranta [Se-99] and formulate a few open problems. For the case of the unit ball, both metrics coincide with the hyperbolic metric. For other domains they are quite different: while δ_G is always a metric, for domains with small boundary α_G may only be a pseudometric. The Apollonian metric was introduced in 1934 by D. Barbilian [Ba, BS], but forgotten for many years. A. Beardon [Be-98] rediscovered it independently in 1998 and thereafter it has been studied very intensively by many authors: see, e.g., Z. Ibragimov [I], P. Hästö [Ha-03, Ha-04, Ha-05, Ha-04, HI-05, HPS-06, HKSV-09], S. Ponnusamy [HPWS-09, HPWW-10], S. Sahoo [SA]. See also D. Herron, W. Ma and D. Minda [HMM].

6.1. Apollonian metric of $G \subsetneq \mathbb{R}^n$.

$$\alpha_G(x, y) = \sup\{\log |a, x, y, b| : a, b \in \partial G\}.$$

- α_G agrees with ρ_G , if G equals B^n and H^n .
- $\alpha_{hG}(hx, hy) = \alpha_G(x, y)$ for $h \in \mathcal{GM}(\mathbb{R}^n)$
- α_G is a pseudometric if ∂G is "degenerate"

6.1.1. Facts.

- (1) The well-known sharp relations $\alpha_G \leq 2j_G$ and $\alpha_G \leq 2k_G$ are due to Beardon [Be-98].
- (2) α_G does not have geodesics.
- (3) The inner metric of the Apollonian metric is called the Apollonian inner metric and it is denoted by $\tilde{\alpha}_G$ (see [Ha-03, Ha-04, HPS-06]).
- (4) We have $\alpha_G \leq \tilde{\alpha}_G \leq 2k_G$.
- (5) $\tilde{\alpha}_G$ -geodesic exists between any pair of points in $G \subsetneq \mathbb{R}^n$ if G^c is not contained in a hyperplane [Ha-04].

6.2. A Möbius invariant metric δ_G . For $x, y \in G \subsetneq \mathbb{R}^n$, Seittenranta [Se-99] introduced the following metric

$$\delta_G(x, y) = \sup_{a, b \in \partial G} \log\{1 + |a, x, b, y|\}.$$

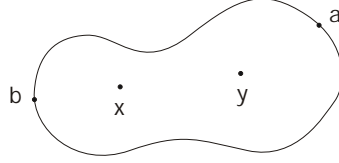


FIGURE 8. A quadruple of points admissible for the definition of the Apollonian metric.

6.2.1. *Facts.* [Se-99]

- (1) The function δ_G is a metric.
- (2) δ_G agrees with ρ_G , if G equals B^n or H^n .
- (3) It follows from the definitions that $\delta_{\mathbb{R}^n \setminus \{a\}} = j_{\mathbb{R}^n \setminus \{a\}}$ for all $a \in \mathbb{R}^n$.
- (4) $\alpha_G \leq \delta_G \leq \log(e^{\alpha_G} + 2) \leq \alpha_G + 3$. The first two inequalities are best possible for δ_G in terms of α_G only [Se-99].

6.3. **Open problem.** Define

$$m_{B^n}(x, y) := 2 \log \left(1 + \frac{|x - y|}{2 \min\{d(x), d(y)\}} \right).$$

Then $m_{B^n}(x, y)$ is not a metric. In fact, any choice of the points on a radial segment will violate the triangle inequality. It is easy to see that $k_{B^n}(x, -x) = m_{B^n}(x, -x)$. We do not know whether $k_{B^n}(x, y) \leq m_{B^n}(x, y)$ for all $x, y \in B^n$. If the inequality holds, then certainly $k_{B^n} \leq 2m_{B^n} \leq 2j_{B^n}$.

6.4. Diameter problems. There exists a domain $G \subsetneq \mathbb{R}^n$ and $x \in G$ such that $j(\partial B_j(x, M)) \neq 2M$ for all $M > 0$. Indeed, let $G = B^n$. Choose $x \in (0, e_1)$ and consider the j -sphere $\partial B_j(0, M)$ for $M = j_G(x, 0)$. Now, $B_j(0, M)$ is a Euclidean ball with radius $|x| = 1 - e^{-M}$. The diameter of the j -sphere $\partial B_j(0, M)$ is

$$j_G(x, -x) = \log \left(1 + \frac{|2x|}{d(x)} \right) = \log \left(1 + \frac{2 - 2e^{-M}}{e^{-M}} \right) = \log(2e^M - 1).$$

We are interested in knowing whether $j_G(x, -x) = 2M$ holds, equivalently in this case, $(e^M - 1)^2 = 0$ which is not true for any $M > 0$. Therefore, we always have $j_G(x, -x) < 2M$ and the diameter of $\partial B_j(0, M)$ is less than twice the radius M . Note that there is no geodesic of the j_G metric joining x and $-x$.

For a convex domain G , it is known by Martio and Väisälä [MV-08] that $k(\partial B_k(x, M)) = 2M$. However, we have the following open problem.

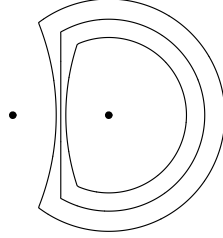


FIGURE 9. Boundaries (nonsmooth!) of j -disks $B_{j_{\mathbb{R}^2 \setminus \{0\}}}(x, M)$ with radii $M = -0.1 + \log 2$, $M = \log 2$ and $M = 0.1 + \log 2$.

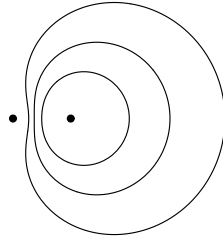


FIGURE 10. Boundaries of quasihyperbolic disks $B_{k_{\mathbb{R}^2 \setminus \{0\}}}(x, M)$ with radii $M = 0.7$, $M = 1.0$ and $M = 1.3$.

6.5. Open problem. Does there exist a number $M_0 > 0$ such that for all $M \in (0, M_0]$ we have $k(\partial B_k(x, M)) = 2M$. For the case of plane domains, this problem was studied by Beardon and Minda [BM-11].

6.6. Convexity problem [Vu-05]. Fix a domain $G \subsetneq \mathbb{R}^n$ and neohyperbolic metric m in a collection of metrics (e.g. quasihyperbolic, Apollonian, j_G , hyperbolic metric of a plane domain etc.). Does there exist constant $T_0 > 0$ such that the ball $B_m(x, T) = \{z \in G : m(x, z) < T\}$, is convex (in Euclidean geometry) for all $T \in (0, T_0)$?

6.7. Theorem. [[Kle-08]] For a domain $G \subsetneq \mathbb{R}^n$ and $x \in G$ the j -balls $B_j(x, M)$ are convex if and only if $M \in (0, \log 2]$.

6.8. Theorem. [[Kle-07], [MO-86]] For $x \in \mathbb{R}^2 \setminus \{0\}$ the quasihyperbolic disk $B_k(x, M)$ is strictly convex iff $M \in (0, 1]$.

Some of the convexity results of Klén have been extended to Banach spaces by A. Rasila and J. Talponen [RT-10]. See also [KRT].

If a metric space is geodesic, then all metric balls are connected. For non-geodesic metric spaces the connectivity of metric balls depends on the setting. For example, chordal balls are always connected while j -balls need not be connected [Kle-08, Remark 4.9 (2)]. See also [KRT].

6.9. Lemma. Let $G \subset \mathbb{R}^n$ be a domain, $x \in G$, and $r > 0$. Then for each connected component D of $B_j(x, r)$ we have

$$\text{diam}_k(D) \leq c(r, n).$$

7. COMPLEMENT OF THE ORIGIN

We have already seen that the quasihyperbolic metric has a simple formula for the complement of the origin. Even more is true: many results of elementary plane geometry hold, possibly with minor modifications, in the quasihyperbolic geometry. Geometrically we can view (G, k_G) , $G = \mathbb{R}^2 \setminus \{0\}$ as a cylindrical surface embedded in \mathbb{R}^3 , cf. [Kle-09].

Therefore many basic results of euclidean geometry hold for (G, k_G) as such or with minor modifications. Some of these results are listed below.

7.1. Theorem. [Law of Cosines] ([Kle-09]) Let $x, y, z \in \mathbb{R}^2 \setminus \{0\}$.

(i) For the quasihyperbolic triangle $\triangle_k(x, y, z)$

$$k(x, y)^2 = k(x, z)^2 + k(y, z)^2 - 2k(x, z)k(y, z) \cos \angle_k(y, z, x).$$

(ii) For the quasihyperbolic trigon $\triangle_k^*(x, y, z)$

$$k(x, y)^2 = k(x, z)^2 + k(y, z)^2 - 2k(y, z)k(z, x) \cos \angle_k(y, z, x) - 4\pi(\pi - \alpha),$$

where $\alpha = \angle(x, 0, y)$.

7.2. Theorem. ([Kle-09]) Let $\triangle_k(x, y, z)$ be a quasihyperbolic triangle. Then the quasihyperbolic area of $\triangle_k(x, y, z)$ is

$$\sqrt{s(s - k(x, y))(s - k(y, z))(s - k(z, x))},$$

where $s = (k(x, y) + k(y, z) + k(z, x))/2$.

It is a natural question to ask whether for some other domains the Law of Cosines holds as an inequality, see [Kle-09]. For the case of the half plane the problem was solved in [HPWW-10].

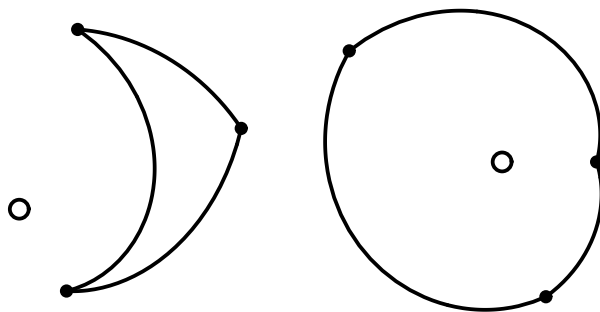


FIGURE 11. An example of a quasihyperbolic triangle (left) and a quasihyperbolic trigon (right). The small circle stands for the puncture at the origin. [Kle-09]

7.3. **Lemma.** ([HPWW-10]) Let $x, y, z \in \mathbb{H}^2$ be distinct points. Then

$$k_{\mathbb{H}^2}(x, y)^2 \geq k_{\mathbb{H}^2}(x, z)^2 + k_{\mathbb{H}^2}(y, z)^2 - 2k_{\mathbb{H}^2}(y, z)k_{\mathbb{H}^2}(x, z) \cos \gamma,$$

where γ is the Euclidean angle between geodesics $J_k[z, x]$ and $J_k[z, y]$.

These results raise many questions about generalizations to more general situations. For instance, what about the case of domains with finitely many boundary points or simple domains such as a sector, a strip or a polygon?

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